Direct Proof

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Introduction

A *direct proof* is one of the most familiar forms of proof. We use it to prove statements of the form "if p then q" or "p implies q" which we can write as $p \Rightarrow q$. The method of the proof is to takes an original statement p, which we assume to be true, and use it to show directly that another statement q is true. So a direct proof has the following steps:

- Assume the statement p is true.
- Use what we know about p and other facts as necessary to deduce that another statement q is true, that is show $p \Rightarrow q$ is true.

Example

Directly prove that if n is an odd integer then n^2 is also an odd integer.

Solution

Let p be the statement that n is an odd integer and q be the statement that n^2 is an odd integer. Assume that n is an odd integer, then by definition $n = 2k + 1$ for some integer k. We will now use this to show that n^2 is also an odd integer.

$$\begin{aligned}
n^2 &= (2k+1)^2 && \text{since } n = 2k+1 \\
&= (2k+1)(2k+1) && \\
&= 4k^2 + 2k + 2k + 1 && \text{by expanding the brackets} \\
&= 4k^2 + 4k + 1 && \\
&= 2(2k^2 + 2k) + 1 && \text{since 2 is a common factor.}
\end{aligned}$$

Hence we have shown that n^2 has the form of an odd integer since $2k^2 + 2k$ is an integer. Therefore we have shown that $p \Rightarrow q$ and so we have completed our proof.

Example

Let a, b and c be integers, directly prove that if a divides b and a divides c then a also divides $b + c$.

Solution

Let a, b and c be integers and assume that a divides b and a divides c. Then as a divides b, by definition, there is some integer k such that $b = ak$. Also as a divides c, by definition, there is some integer l such that $c = al$. Note that we use different letters k and l to stand for the integers

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because we do not know if b and c are equal or not. We will now use these two facts to get our conclusion. So

$$\begin{aligned} b + c &= (ak) + (al) && \text{by our definitions of } b \text{ and } c \\ &= a(k + l) && \text{since } a \text{ is a common factor.} \end{aligned}$$

Hence a divides $b + c$ since $k + l$ is an integer.

Example

Directly prove that if m and n are odd integers then mn is also an odd integer.

Solution

Assume that m and n are odd integers. Then by definition $m = 2k + 1$ for some integer k and $n = 2l + 1$ for some integer l. Again note that we have used different integers k and l in the definitions of m and n. We will now use this to show that mn is also an odd integer.

$$\begin{aligned} mn &= (2k + 1)(2l + 1) && \text{by our definitions of } m \text{ and } n \\ &= 4kl + 2k + 2l + 1 && \text{by expanding the brackets} \\ &= 2(2kl + k + l) + 1 && \text{since 2 is a common factor.} \end{aligned}$$

Hence we have shown that mn has the form of an odd integer since $2kl + k + l$ is an integer.

Example

Let m and n be integers. Directly prove that if m and n are perfect squares then mn is also a perfect square.

Solution

Recall the definition that an integer m is a perfect square if $m = k^2$ for some integer k. Now assume that m and n are integers and are perfect squares. Then by definition $m = k^2$ for some integer k and $n = l^2$ for some integer l. We will now use these facts to show that mn is also a perfect square.

$$mn = k^2 l^2 = (kl)^2$$

and kl is an integer, therefore mn is a perfect square.

Exercises

Prove directly that

1. If n is an even integer then $7n + 4$ is an even integer.
2. If m is an even integer and n is an odd integer then $m + n$ is an odd integer.
3. If m is an even integer and n is an odd integer then mn is an even integer.
4. If a, b and c are integers such that a divides b and b divides c then a divides c.

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