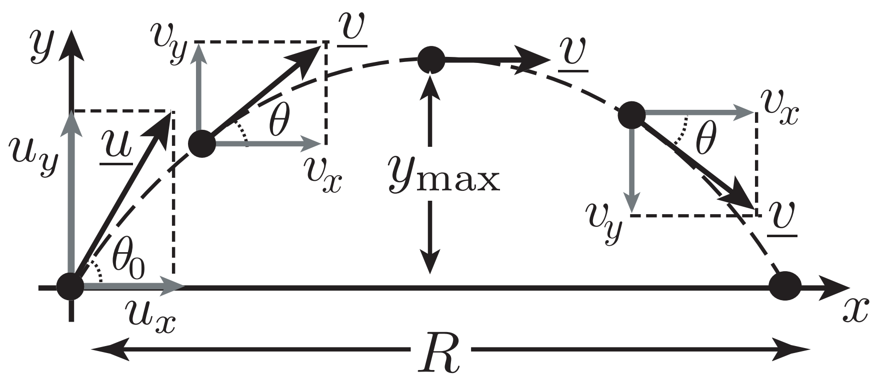
7. Motion in a Plane: Projectiles

Any object that is given an initial velocity and which subsequently follows a path determined by the gravitational force acting on it and by the frictional resistance of the atmosphere is called a projectile.



Consider a body projected from the origin $(0, 0)$ with initial velocity $\underline{u} = (u_x, u_y)$ at an angle of departure θ_0. At any later time t, let (x, y) be its coordinates, and $\underline{v} = (v_x, v_y)$ its velocity. θ is the angle $\underline{v}$ makes with the horizontal, measured in an anti-clockwise sense. If we neglect air resistance, the motion of the projectile can be described as a combination of horizontal motion with constant velocity and vertical motion with constant acceleration. This follows from Newton's Second Law which, in component form, gives

$$\frac{dv_x}{dt} = 0 \text{ and so } v_x = u_x = u \cos\theta_0$$

$$\frac{dv_y}{dt} = -g \text{ and so } v_y = u_y - gt = u \sin\theta_0 - gt$$

The speed v and angle θ are then given by

$$v = \sqrt{v_x^2 + v_y^2} \qquad \tan\theta = \frac{v_y}{v_x}$$

The coordinates of the projectile are

$$x = u_x t = (u \cos\theta_0)t$$

$$y = u_y t - \frac{1}{2}gt^2 = (u \sin\theta_0)t - \frac{1}{2}gt^2$$

The two preceding equations give the equation of the trajectory in terms of the parameter t. By eliminating t, the equation in terms of x and y is

$$y = (\tan\theta_0)x - \frac{g}{2u^2\cos^2\theta_0}x^2$$

This last equation can be recognised as the equation of a parabola. At the highest point, the vertical velocity, v_y, is zero, and hence the time to reach the highest point is $\dfrac{u\sin\theta_0}{g}$.

The highest point is given by $y_{\max} = \dfrac{u^2\sin^2\theta_0}{2g}$.

The horizontal range, R, is the horizontal distance from the starting point to the point at which the projectile returns to its original elevation, and at which therefore $y = 0$. Hence $R = \dfrac{u^2\sin 2\theta_0}{g}$.

The maximum range occurs when $\sin 2\theta_0 = 1$, i.e. when $\theta_0 = \dfrac{\pi}{4}$ and then $R_{\max} = \dfrac{u^2}{g}$.