9. Motion of a particle (2)

Circular motion:

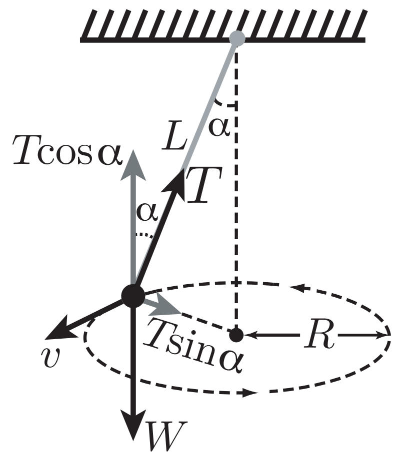
In circular motion, r is constant and so $\dot{r} = \ddot{r} = 0$. The velocity and acceleration vectors are then

$$\dot{\underline{r}} = r\dot{\theta}\underline{e}_\theta \qquad \ddot{\underline{r}} = -r\dot{\theta}^2\underline{e}_r + r\ddot{\theta}\underline{e}_\theta$$

When the circular motion is uniform the speed, $r\dot{\theta}$, is constant, v say, and so $\ddot{\theta} = 0$. Then $\dot{\underline{r}} = v\underline{e}_\theta$, $\ddot{\underline{r}} = -\dfrac{v^2}{r}\underline{e}_r$. So, if a particle of mass m moves uniformly in a circle of radius r, with speed v, the radial acceleration has magnitude v^2/r and is directed inward along the radius.

The conical pendulum:

A particle of mass m revolves in a horizontal circle with constant speed v at the end of a cord of length L. The cord makes an angle α with the vertical. The radius of the circle is $R = L\sin\alpha$. Hence $v = (L\sin\alpha)\omega$, where $\omega = \dot{\theta}$ is the angular speed of motion in the horizontal circle.



The forces exerted on the body are its weight, of magnitude W, and the tension in the cord which resolves into horizontal and vertical components of magnitudes $T\sin\alpha$ and $T\cos\alpha$ resp. The body has no vertical acceleration and the radial acceleration has magnitude v^2/R.

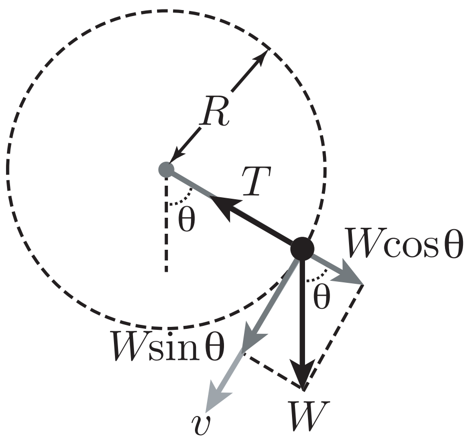
From Newton's 2nd law vertically and radially

$$T\cos\alpha - W = 0 \quad \text{and} \quad T\sin\alpha = \frac{mv^2}{R}$$

Then $\tan\alpha = \dfrac{v^2}{Rg}$ and $\cos\alpha = \dfrac{g/L}{\omega^2}$. Motion arises only if $\cos\alpha < 1$, that is $\omega^2 > g/L$. If $\omega^2 < g/L$ then $\alpha = 0$.

Motion in a vertical circle:

Consider a small body of mass m attached to a cord of length R and whirling in a vertical circle about O. The cord makes an angle θ, measured anti-clockwise from the downward vertical. The motion is circular but not uniform. The forces acting on the body are its weight, $\underline{W} = m\underline{g}$, and the tension $\underline{T}$ in the cord. The radial acceleration has magnitude v^2/R where $v = R\dot{\theta}$ (not necessarily constant).



From Newton's 2nd law in the radial direction the magnitude of the tension in the cord is $T = m\left(\dfrac{v^2}{R} + g\cos\theta\right)$. From Newton's 2nd law in the tangential direction $-mg\sin\theta = mR\ddot{\theta}$.

Writing $\dot{\theta}$ as ω, $\dfrac{d^2\theta}{dt^2} = \dfrac{d\omega}{dt} = \omega\dfrac{d\omega}{d\theta}$, this becomes $-mg\sin\theta = mR\omega\dfrac{d\omega}{d\theta}$ which by integrating gives the energy equation $\frac{1}{2}mV^2 = \frac{1}{2}mv^2 + mgR(1-\cos\theta)$ where V is the speed when $\theta = 0$. The critical speed below which the cord becomes slack ($T = 0$) at its highest point (where $\theta = \pi$) is $v_c = \sqrt{Rg}$.