8. Motion of a particle (1)

If a moving particle P has cartesian coordinates (x, y) its position vector is $\underline{r} = x\underline{i} + y\underline{j}$ where both x and y are functions of time, t. Since $\underline{i}$ and $\underline{j}$ are constant vectors, it follows, by differentiating, that its velocity and acceleration vectors are $\underline{v} = \dot{\underline{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$ and $\underline{a} = \ddot{\underline{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$. Here the dot $\cdot$ denotes a derivative with respect to t. In polar coordinates (r, θ), $x = r\cos\theta$ and $y = r\sin\theta$. Define unit vectors radially and tangentially as $\underline{e}_r$ and $\underline{e}_\theta$. Then

$$\underline{e}_r = \cos\theta\underline{i} + \sin\theta\underline{j} \qquad \underline{e}_\theta = -\sin\theta\underline{i} + \cos\theta\underline{j}$$

$$\dot{\underline{e}}_r = -\sin\theta\,\dot{\theta}\underline{i} + \cos\theta\,\dot{\theta}\underline{j} = \dot{\theta}\,\underline{e}_\theta$$

$$\dot{\underline{e}}_\theta = -\cos\theta\,\dot{\theta}\underline{i} - \sin\theta\,\dot{\theta}\underline{j} = -\dot{\theta}\,\underline{e}_r$$

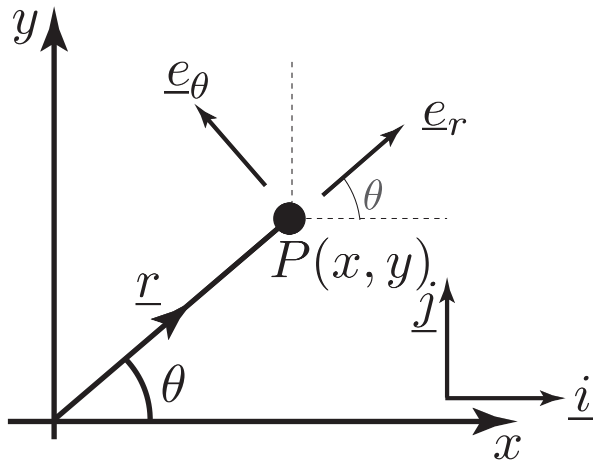
It then follows that

$$\underline{r} = r\underline{e}_r$$

$$\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

$\dot{\theta}$ is the **angular velocity**, ω.



Simple Harmonic Motion (SHM):

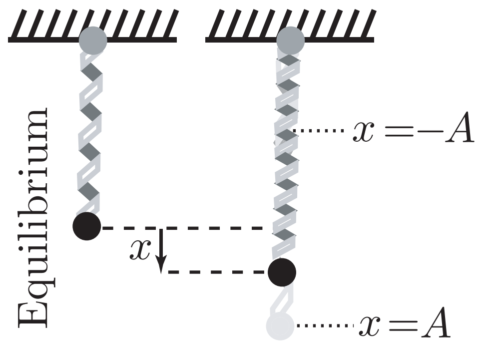
The motion of a body under the influence of a restoring force proportional to displacement is called **Simple Harmonic Motion** (SHM).

For the one-dimensional motion of a point mass, m, as in a mass-spring oscillator, the force is given by $-kx$, where k is the spring constant and x is the displacement from equilibrium; the equation of SHM is

$$-kx = m\frac{d^2x}{dt^2}$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

where $\omega^2 = k/m$.



The solution of this equation is

$$x(t) = C\cos\omega t + D\sin\omega t = A\cos(\omega t + \epsilon)$$

for arbitrary constants C and D, and so

$$v(t) = -\omega A\sin(\omega t + \epsilon)$$

It follows that $v^2 = \omega^2(A^2 - x^2)$. Here t is time, the amplitude, A, is the maximum value of $|x|$, v is the velocity, and ϵ is the initial phase angle. The period, τ, is the time for a complete oscillation. The frequency, f, is the number of oscillations per unit time. ω is the angular frequency given by $\omega = \dfrac{2\pi}{\tau} = 2\pi f = \sqrt{\dfrac{k}{m}}$. The graph shows $x(t)$ for the case $\epsilon = 0$. The initial position of the particle is its maximum positive displacement.

The maximum speed occurs when $x = 0$, i.e. at the centre of the oscillation. The acceleration is maximum when the displacement x is maximum.