# Parametric Differentiation

mc-TY-parametric-2009-1

Instead of a function $y(x)$ being defined explicitly in terms of the independent variable $x$, it is sometimes useful to define both $x$ and $y$ in terms of a third variable, $t$ say, known as a parameter. In this unit we explain how such functions can be differentiated using a process known as parametric differentiation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate a function defined parametrically
- find the second derivative of such a function

## Contents



 

# 1. Introduction

Some relationships between two quantities or variables are so complicated that we sometimes introduce a third quantity or variable in order to make things easier to handle. In mathematics this third quantity is called a **parameter**. Instead of one equation relating say, $x$ and $y$, we have two equations, one relating $x$ with the parameter, and one relating $y$ with the parameter. In this unit we will give examples of curves which are defined in this way, and explain how their rates of change can be found using parametric differentiation.

# 2. The parametric definition of a curve

In the first example below we shall show how the $x$ and $y$ coordinates of points on a curve can be defined in terms of a third variable, $t$, the **parameter**.

**Example**

Consider the parametric equations

$$x = \cos t \qquad y = \sin t \qquad \text{for } 0 \le t \le 2\pi \tag{1}$$

Note how both $x$ and $y$ are given in terms of the third variable $t$.

To assist us in plotting a graph of this curve we have also plotted graphs of $\cos t$ and $\sin t$ in Figure 1. Clearly,

when $t = 0$, $x = \cos 0 = 1$; $y = \sin 0 = 0$

when $t = \frac{\pi}{2}$, $x = \cos \frac{\pi}{2}$=0; $y = \sin \frac{\pi}{2} = 1$.

In this way we can obtain the $x$ and $y$ coordinates of lots of points given by Equations (1). Some of these are given in Table 1.

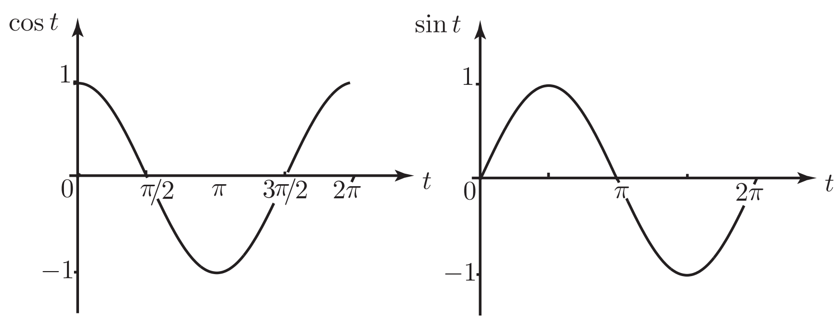


Figure 1. Graphs of $\sin t$ and $\cos t$.

| $t$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $x$ | 1 | 0 | −1 | 0 | 1 |
| $y$ | 0 | 1 | 0 | −1 | 0 |

Table 1. Values of $x$ and $y$ given by Equations (1).

 

Plotting the points given by the $x$ and $y$ coordinates in Table 1, and joining them with a smooth curve we can obtain the graph. In practice you may need to plot several more points before you can be confident of the shape of the curve. We have done this and the result is shown in Figure 2.

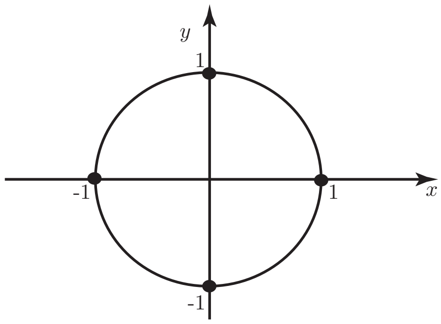


Figure 2. The parametric equations define a circle centered at the origin and having radius 1.

So $x = \cos t$, $y = \sin t$, for $t$ lying between 0 and $2\pi$, are the parametric equations which describe a circle, centre $(0, 0)$ and radius 1.

# 3. Differentiation of a function defined parametrically

It is often necessary to find the rate of change of a function defined parametrically; that is, we want to calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$. The following example will show how this is achieved.

**Example**

Suppose we wish to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = \cos t$ and $y = \sin t$.

We differentiate both $x$ and $y$ with respect to the parameter, $t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$$

From the chain rule we know that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

so that, by rearrangement

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \text{ is not equal to } 0$$

So, in this case

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\cos t}{-\sin t} = -\cot t$$

 

**Key Point**

parametric differentiation: if $x = x(t)$ and $y = y(t)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

**Example**

Suppose we wish to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = t^3 - t$ and $y = 4 - t^2$.

$$x = t^3 - t \qquad\qquad y = 4 - t^2$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 1 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2t$$

From the chain rule we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-2t}{3t^2 - 1} \end{aligned}$$

So, we have found the gradient function, or derivative, of the curve using parametric differentiation.
For completeness, a graph of this curve is shown in Figure 3.

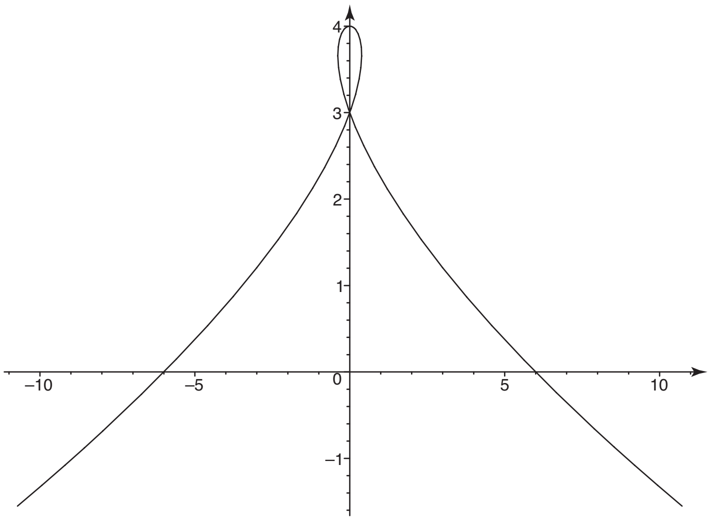


Figure 3

**Example**

Suppose we wish to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = t^3$ and $y = t^2 - t$.

In this Example we shall plot a graph of the curve for values of $t$ between $-2$ and $2$ by first producing a table of values (Table 2).

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $x$ | $-8$ | $-1$ | $0$ | $1$ | $8$ |
| $y$ | $6$ | $2$ | $0$ | $0$ | $2$ |

Table 2

Part of the curve is shown in Figure 4. It looks as though there may be a turning point between 0 and 1. We can explore this further using parametric differentiation.

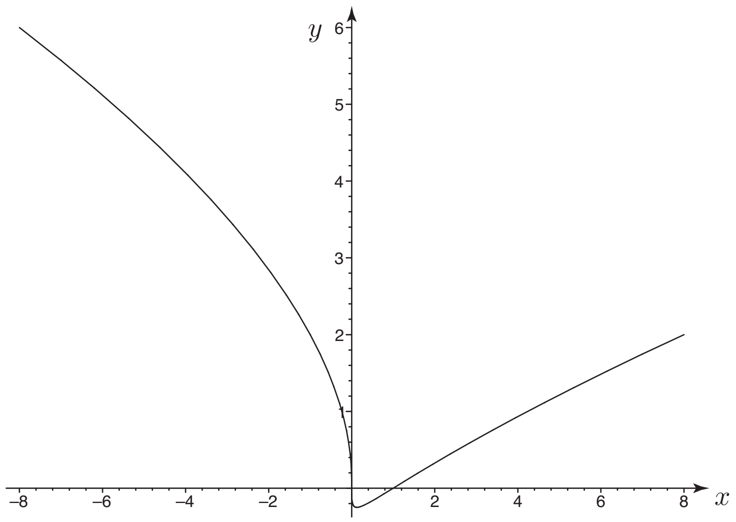


Figure 4.

From

$$x = t^3 \qquad\qquad y = t^2 - t$$

we differentiate with respect to $t$ to produce

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t - 1$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided} \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-1}{3t^2}$$

From this we can see that when $t = \frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so $t = \frac{1}{2}$ is a stationary value. When $t = \frac{1}{2}$, $x = \frac{1}{8}$ and $y = -\frac{1}{4}$ and these are the coordinates of the stationary point.

We also note that when $t = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is infinite and so the $y$ axis is tangent to the curve at the point $(0, 0)$.

**Exercises 1**

1. For each of the following functions determine $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

   (a) $x = t^2 + 1, y = t^3 - 1$
   (b) $x = 3\cos t, y = 3\sin t$
   (c) $x = t + \sqrt{t}, y = t - \sqrt{t}$
   (d) $x = 2t^3 + 1, y = t^2 \cos t$
   (e) $x = t\mathrm{e}^{-t}, y = 2t^2 + 1$

2. Determine the co-ordinates of the stationary points of each of the following functions

   (a) $x = 2t^3 + 1, y = t\mathrm{e}^{-2t}$
   (b) $x = \sqrt{t} + 1, y = t^3 - 12t$ for $t > 0$
   (c) $x = 5t^4, y = 5t^6 - t^5$ for $t > 0$
   (d) $x = t + t^2, y = \sin t$ for $0 < t < \pi$
   (e) $x = t\mathrm{e}^{2t}, y = t^2\mathrm{e}^{-t}$ for $t > 0$

# 4. Second derivatives

**Example**

Suppose we wish to find the second derivative $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when

$$x = t^2 \qquad y = t^3$$

Differentiating we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2}{2t} = \frac{3t}{2}$$

We can apply the chain rule a second time in order to find the second derivative, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{\frac{3}{2}}{2t} \\
&= \frac{3}{4t}
\end{aligned}$$

 

### Key Point

if $x = x(t)$ and $y = y(t)$ then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

**Example**

Suppose we wish to find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when

$$x = t^3 + 3t^2 \qquad y = t^4 - 8t^2$$

Differentiating

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 + 6t \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3 - 16t$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4t^3 - 16t}{3t^2 + 6t}$$

This can be simplified as follows

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4t(t^2 - 4)}{3t(t + 2)} \\ &= \frac{4t(t + 2)(t - 2)}{3t(t + 2)} \\ &= \frac{4(t - 2)}{3} \end{aligned}$$

We can apply the chain rule a second time in order to find the second derivative, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\frac{4}{3}}{3t^2 + 6t} \\ &= \frac{4}{9t(t + 2)} \end{aligned}$$

 

## Exercises 2

For each of the following functions determine $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of $t$

1. $x = \sin t, y = \cos t$
2. $x = 3t^2 + 1, y = t^3 - 2t^2$
3. $x = \dfrac{1}{2}t^2 + 2, y = \sin(t+1)$
4. $x = \mathrm{e}^{-t}, y = t^3 + t + 1$
5. $x = 3t^2 + 4t, y = \sin 2t$

## Answers

### Exercise 1

1. a) $\dfrac{3t}{2}$ b) $-\cot t$ c) $\dfrac{2\sqrt{t}-1}{2\sqrt{t}+1}$ d) $\dfrac{2\cos t - t\sin t}{6t}$ e) $\dfrac{4t\mathrm{e}^t}{1-t}$

2. a) $\left(\dfrac{5}{4}, \dfrac{1}{2\mathrm{e}}\right)$ b) $(1+\sqrt{2}, -16)$ c) $\left(\dfrac{5}{1296}, \dfrac{-1}{46656}\right)$ d) $\left(\dfrac{\pi}{2} + \dfrac{\pi^2}{4}, 1\right)$ e) $(2\mathrm{e}^4, 4\mathrm{e}^{-2})$

### Exercise 2

1. $-\sec^3 t$ 2. $\dfrac{1}{12t}$ 3. $\dfrac{-t\sin(t+1) - \cos(t+1)}{t^3}$

4. $(3t^2 + 6t + 1)\mathrm{e}^{2t}$ 5. $\dfrac{-2(3t+2)\sin 2t - 3\cos 3t}{2(3t+2)^3}$

 